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Continuous feedback approach for controlling chaos

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We show that the *continuous* feedback approach is highly effective for controlling chaotic systems. The control design for the Lorenz system is presented as an example to demonstrate the strength of this approach. The proposed control is able to eliminate chaos and bring the system toward any of the three steady states. Two different control input locations are considered. Only one system variable is used in the feedback. The control scheme can tolerate both measurement noise and modeling uncertainty as long as they are bounded.

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The problem of controlling chaotic systems has attracted great attention and interest in recent years [1–8]. It has been demonstrated experimentally [2–4] that it is possible to stabilize periodic orbits by applying *intermittent* perturbations to the system. This is based on the fact that a chaotic attractor usually has an infinite number of unstable periodic orbits embedded in it. Therefore, a carefully chosen perturbation is able to stabilize some of these unstable orbits [1]. In this paper, we approach the problem from a different direction. Assume that the model equations of the system, which may contain some uncertainty, are available. We will show that the use of a *continuous* feedback approach is highly effective for controlling a chaotic system. The resulting system performance is studied by rigorous analysis. We shall use the Lorenz equation [9] as an example to illustrate this approach. Several previous experimental and theoretical studies on controlling this system have been reported [6,8]. However, nonlinearity was not fully addressed in some analyses. This paper will show that the control can drive the chaotic system to any of the three (stable or unstable) steady states, independent of initial conditions. Furthermore, this control is effective even in the presence of bounded measurement noise and uncertain parameter variations.

This work is motivated by the following considerations: in a physical system that exhibits chaos, it is always desirable to establish the model dynamical equations by invoking fundamental physical laws; one may anticipate uncertainty in the modeling due to imperfect knowledge of the system; if model equations can be found, it is advantageous to exploit this knowledge to manipulate or control system performance; a practical control scheme should be sufficiently reliable that uncertainties in the feedback signal (i.e., measurement noise) and/or the physical model will not cause major changes in performance; finally, the control design should be flexible so that if the desired system performance changes, the control can be easily modified to accommodate the change.

The control design procedure follows two steps. First, determine the physical form of the controlling signal (namely, the control location in the system equations). This can be achieved by identifying the channel through which (external) energy is transmitted to the system or other influential parameters that may affect the system. The system with control is in the phase space form:

$$\dot{X} = f(X,u,t) \, ,$$

where t is the time, X is the phase vector, u is the control, and $f(\ )$ is the mapping that describes the system under control. Second, design a feedback control scheme $u = u(X,t)$ to render a specific system performance. For example, if stability around a preselected steady position is desired, choose a Lyapunov function $v > 0$ and a control u such that $\dot{v} < 0$ along the trajectory of the controlled system. Sensors are used to measure the current state of the system. The control signal u is determined based on on-line computation.

The Lorenz equation [9] is used to illustrate the advantages of the continuous feedback approach because the Lorenz system is one of the most studied chaotic systems. Researchers can easily gain insight into the design and compare system performance with and without the use of control. The system representation is also, in a sense, generic. It has been proposed that the same set of equations may be used to model various different physical systems [10,11]. Furthermore, tolerating modeling uncertainty is particularly important for this system since, in some cases, the Lorenz system is a simplified (or truncated) model of the physical setting.

Consider the following Lorenz system with an external control u added in the second equation:

$$\dot{x}=\sigma(y-x)\ ,$$
$$\dot{y}=rx-y-xz+u\ , \tag{1}$$
$$\dot{z}=xy-bz\ .$$

The physical meanings of x, y, z, σ, r, and b depend on the particular system. This control location has physical implications and can be experimentally implemented. For example, in the toroidal thermal convection loop setting, u corresponds to the asymmetric perturbation to the wall temperature [6], or the change of the loop tilt angle from the vertical [11].

We will address the control problem of driving the system, specified by (x,y,z) to any of the three equilibrium positions of the uncontrolled (i.e., $u=0$) system: $(\pm\beta,\pm\beta,r-1)$ and (0,0,0), where $\beta:=[b(r-1)]^{1/2}$. We assume $r>1$ in this work since otherwise only one equilibrium position (0,0,0) exists, which is the trivial case.

We shall first consider the control design for the equilibrium position $(\beta,\beta,r-1)$. The design for the other two cases is similar. Define the new state variables $x_1:=x-\beta$, $x_2:=y-\beta$, $x_3:=z-(r-1)$. The system in Eq. (1) can be transformed to

$$\dot{x}_1=\sigma(x_2-x_1)\ , \tag{2a}$$

$$\dot{x}_2=x_1-x_2-x_1x_3-\beta x_3+u\ , \tag{2b}$$

$$\dot{x}_3=x_1x_2-bx_3+\beta(x_1+x_2)\ . \tag{2c}$$

Let $X:=(x_1,x_2,x_3)^T$. Driving the state (x,y,z) to $(\beta,\beta,r-1)$ is equivalent to driving X to 0. We propose the following linear continuous feedback control scheme:

$$u=-kx_1\ , \tag{3}$$

where k, the design parameter, is to be chosen later. The physical motivation for this control scheme is clear. The choice of u directly reflects the design intention: if x_1 is far from 0, then the control action will be large to draw the system to the target. If x_1 is close to 0, then the control action will be small to fine tune the system performance.

The resulting controlled system is given by Eqs. (2a) and (2c), and

$$\dot{x}_2=-(k-1)x_1-x_2-x_1x_3-\beta x_3\ . \tag{4}$$

The controlled system is continuous in X and hence a solution exists. We will prove that $X=0$ is globally asymptotically stable and that X converges to 0 at least exponentially if the system parameters are constant and known. To prove stability, choose the Lyapunov function candidate,

$$v=\tfrac{1}{2}\{[(k-1)/\sigma]x_1^2+x_2^2+x_3^2\}\ . \tag{5}$$

Taking the derivative of v along the trajectory of the controlled system yields

$$\dot{v}=-[x_1\ \ x_3]\begin{bmatrix}k-1 & -\frac{1}{2}\beta\\ -\frac{1}{2}\beta & b\end{bmatrix}\begin{bmatrix}x_1\\ x_3\end{bmatrix}-x_2^2\ . \tag{6}$$

If one chooses

$$k>(\beta^2/4b)+1\ , \tag{7}$$

then the 2×2 matrix (say, Ψ) in Eq. (6) is positive definite. This in turns implies that

$$\dot{v}\leq-\lambda_3\|X\|^2\ , \tag{8}$$

where $0<\lambda_3:=\min\{\lambda_{\min}(\Psi),1\}$ and $\lambda_{\min}(\Psi)$ is the minimum eigenvalue of Ψ. Furthermore, from Eq. (5) we have $\lambda_1\|X\|^2\leq v\leq\lambda_2\|X\|^2$, where $0<\lambda_1:=\frac{1}{2}\min\{(k-1)/\sigma,1\}$ and $0<\lambda_2:=\frac{1}{2}\max\{(k-1)/\sigma,1\}$. Global asymptotic stability of $X=0$ is proved. Next, we shall show that X converges to 0 at least exponentially. Using Eq. (8) yields $\dot{v}\leq-(\lambda_3/\lambda_2)v$ and hence, upon invoking the theory of differential inequalities [12], $v\leq v_0\exp[-(\lambda_3/\lambda_2)(t-t_0)]$, where $v_0=v[X(t_0)]$ and t_0 is the initial time. This in turn implies

$$\|X(t)\|\leq\left[\frac{v_0}{\lambda_1}\right]^{1/2}\exp\left[-\frac{\lambda_3}{2\lambda_2}(t-t_0)\right]\ . \tag{9}$$

It is possible in some cases that one cannot find the suitable control scheme directly by intuition or physical arguments since chaotic models are often complicated, nonlinear, and multidimensional. The dynamic characteristics are not always obvious. However, one is often able to work backward and find the feedback control scheme under which the desired performance is assured in the analysis. For example, in the above case, one searches for an appropriate control scheme that ensures that the Lyapunov derivative $\dot{v}$ has the desirable properties (e.g., being negative definite). In this way, the designer will in fact make a *constructive* use of the analysis tool, which can also help choose a control scheme that may not be intuitively obvious but is practically effective.

Some researchers may wonder whether one should use *feedback* control for chaotic systems. A typical argument is as follows: since the chaotic system is sensitive to small errors, a small noise in the feedback measurement may result in totally different system performance. First, we wish to point out that the controlled system no longer shares the same dynamic characteristics of the original uncontrolled system; the new dynamics include the control u. Second, we will prove that the current controlled system is *robust* to measurement noise, meaning satisfactory performance is still guaranteed if the measurement noise is small.

Suppose that the measured state is $\hat{x}_1(t)=x_1(t)+w_1(t)$, where $w_1(t)$ is the measurement noise. It is practical to assume that the measurement noise is unknown, time varying, but bounded: $|w_1(t)|\leq\bar{w}_1$ for all t, where $\bar{w}_1(\geq0)$ is a constant. The linear feedback control law used for this situation is the same as Eq. (3), except now that x_1 is replaced by $\hat{x}_1$: $u=-k\hat{x}_1$. We now show that X of Eq. (2) under this control is (globally) *uniformly ultimately bounded* (GUUB). This means that X will enter a sphere, centered at the origin, in the phase space in a finite time and stays there thereafter. Using the Lyapunov function in Eq. (5), it is straightforward to show that

$$\dot{v} = [\text{right hand side of Eq. (6)}] + x_2(-kw_1) . \quad (10)$$

Since $-x_2 k w_1 \le k\bar{w}_1 |x_2| \le k\bar{w}_1 \|X\|$, we have $\dot{v} \le -\lambda_3 \|X\|^2 + k\bar{w}_1 \|X\|$. Hence there exist constants $\delta_{1,2} > 0$, such that $\dot{v} \le -\delta_1$ for all $\|X\| > (k\bar{w}_1/\lambda_3) + \delta_2$. This results in GUUB upon invoking the standard arguments in Ref. [13]. The radius of the sphere and the magnitude of the finite time for GUUB both depend on λ_3 and $\bar{w}_1$. As $\bar{w}_1 \to 0$, the radius $\to 0$. In the extreme case $\bar{w}_1 \equiv 0$, the performance displays asymptotic stability.

Another important issue that must be addressed in control design is modeling uncertainty. Uncertainty often arises from imperfect knowledge of the system or a simplification in the modeling procedure. The control scheme, which is designed based on the (approximate) model, must perform satisfactorily even if the model is not exact.

For example, in Ref. [9] the positive constants σ, r, and b depend on the Prantle and Rayleigh numbers and the physical configuration of the system. It is reasonable to anticipate that these parameters change with time (e.g., due to temperature-induced viscosity change). We shall show that tuning the design parameter k appropriately preserves a satisfactory system performance. To simplify the analysis, we only consider the case when both b and r (hence β) are uncertain, time varying, but bounded: $b_{\min} \le b(t) \le b_{\max}$, $r_{\min} \le r(t) \le r_{\max}$, $\beta_{\min} \le \beta(t) \le \beta_{\max}$. Here "max" and "min" stand for the possible bounds, which are *known*, of the designated parameters. Their "nominal" values are denoted by $\bar{r}$, $\bar{b}$, and $\bar{\beta}$, respectively. The equilibrium positions are in terms of the nominal values [hence $(\pm\bar{\beta}, \pm\bar{\beta}, \bar{r}-1)$]. Define the new state variables $x_1 := x - \bar{\beta}$, $x_2 := y - \bar{\beta}$, and $x_3 := z - (\bar{r}-1)$. The system in Eq. (1) can be transformed to Eq. (2a) and

$$\dot{x}_2 = x_1 - x_2 - x_1 x_3 - \bar{\beta} x_3 + (r - \bar{r}) x_1 + (r\bar{\beta} - \bar{r}\bar{\beta}) + u , \quad (11a)$$

$$\dot{x}_3 = x_1 x_2 - b x_3 + \bar{\beta}(x_1 + x_2) + [\bar{\beta}^2 - b(\bar{r}-1)] . \quad (11b)$$

Using Eqs. (3) and (5), one can show easily that by taking

$$k > (\bar{\beta}^2 / 4b_{\min}) + [(r_{\max} - \bar{r})^2/4] + 1 , \quad (12)$$

there exist constants $\xi_{1,2} > 0$, such that

$$\dot{v} \le -\xi_1 \|X\|^2 + \xi_2 \|X\| . \quad (13)$$

Hence there exist constants $\hat{\delta}_{1,2} > 0$, such that $\dot{v} \le -\hat{\delta}_1$ for all $\|X\| > (\xi_2/\xi_1) + \hat{\delta}_2$. Again this results is GUUB. Finally, we note that both measurement noise and modeling uncertainty issues can be simultaneously addressed by combining the analysis shown above.

The control scheme for the equilibrium position $(-\beta, -\beta, r-1)$ is the same as Eq. (3), except that β is replaced by $-\beta$. The control scheme for the equilibrium position (0,0,0) is given by $u = -2rx$ in Eq. (1). Global asymptotic stability can be proved by taking the time derivative of $v = \frac{1}{2}[(r/\sigma)x^2 + y^2 + z^2]$ along the trajectory of Eq. (1), which yields $\dot{v} = -rx^2 - y^2 - bz^2 < 0$. Again, it can be shown that the control tolerates both measurement noise and modeling uncertainty.

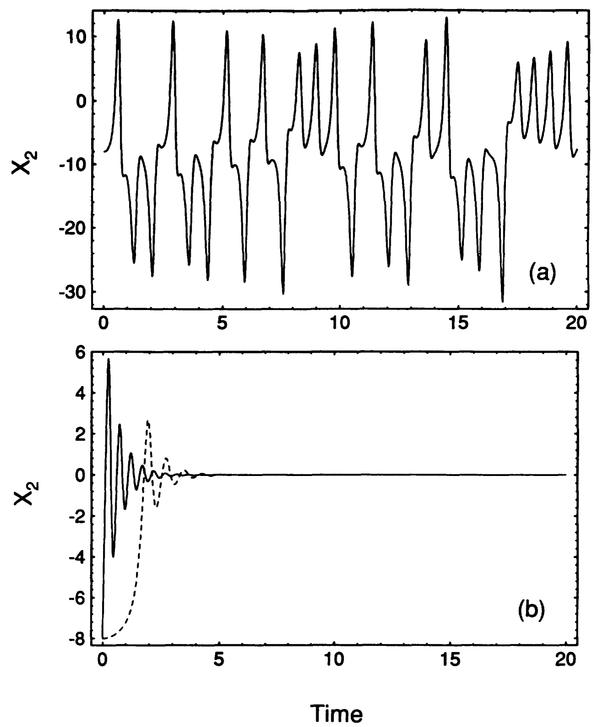

FIG. 1. Trajectory history of $x_2(t)$ in Eq. (2) with $\sigma = 10$, $b = \frac{8}{3}$, $r = 28$, and the initial conditions $x_1(0) = x_2(0) = -8$ and $x_3(0) = 0$. (a) Without using the feedback control. (b) With the feedback controls: Eq. (2) ($k = 10$, solid line) and Eq. (14) (dashed line).

In practice, it is also possible to relocate the control to the third mode:

$$\dot{x}_3 = x_1 x_2 - b x_3 + \beta(x_1 + x_2) + u . \quad (14)$$

Note that this control is also physically implementable. For example, in Ref. [6], u represents the heating rate. Again, design u to drive X toward 0. The feedback control scheme is given by $u = -\beta x_1$. The motivation for this scheme is similar to that of Eq. (3). Consider the Lyapunov function $v = \frac{1}{2}[(1/\sigma)x_1^2 + x_2^2 + x_3^2]$ for all $X \in \Omega$, $\Omega := \{x \mid \|X\| < \sqrt{2}\beta\}$. Its derivative along the trajectory of the controlled system is given by $\dot{v} = -(x_1 - x_2)^2 - b x_3^2$. Since $\dot{v} \le 0$ for all $X \ne 0$, $X = 0$ is stable. Next, since $\dot{v} = 0$ occurs at the set $M = \{x_1 = x_2, x_3 = 0\}$, the largest invariant set within M and Ω is (0,0,0). By La Salle's theorem [14], one concludes that $X(t) \to 0$ as $t \to \infty$. Thus $X = 0$ is locally asymptotically stable.

Computer simulations were performed with $\sigma = 10$, $b = \frac{8}{3}$, and $r = 28$. The trajectory of $x_2(t)$ with and without the feedback control in Eq. (2) or (14) is shown in Fig. 1. Note that in this parameter range the three equilibrium positions of the uncontrolled system are all unstable.

In summary, a continuous feedback approach for controlling chaos is used. Given the system's model dynamic equation, the control is designed for a well-specified system performance. The approach is applied to the Lorenz system and is able to drive the system to any steady state. In addition, the control can tolerate measurement noise and modeling uncertainty. The control design can be easily implemented in real physical systems.

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